Cross-Sectional Conditional Independence in Stationary Time Series

Graphical Equivalence and Completeness of Collider Separation

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CLeaR 2026 Conference

April 2026

novo nordisk foundation

MACHINE LEARNING
UNIVERSITY OF COPENHAGEN

CoCaLab
Copenhagen Causality Lab

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Snapshots

Reality is time evolving

Snapshots

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Biology

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Social sciences

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Past common causes induce contemporaneous dependencies in snapshots!

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Social sciences

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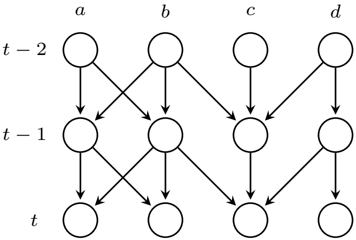
What is the right tool to capture conditional independencies for snapshots?

Graphical separations - a trailer

(a) Space-time DAG



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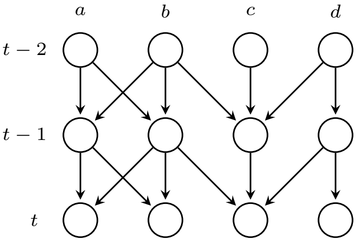

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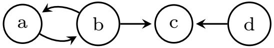

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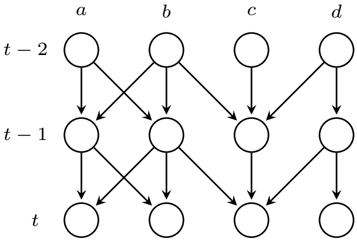

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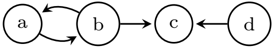

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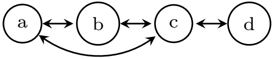

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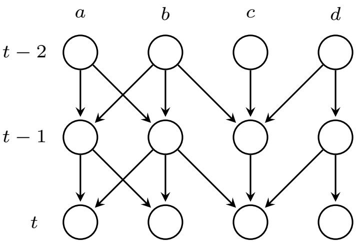

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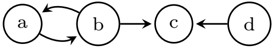

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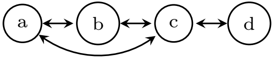

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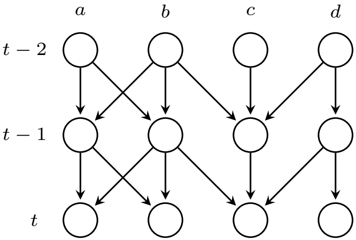

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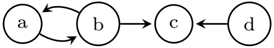

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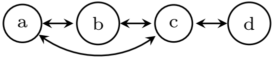

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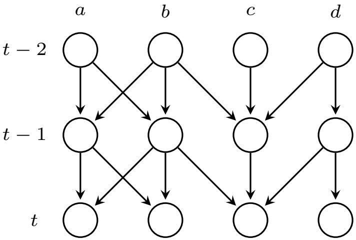

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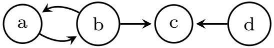

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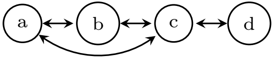

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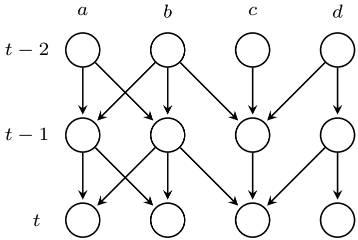

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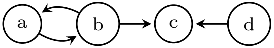

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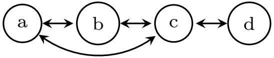

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We show that these two separations are the same!

+ we translate them to space-time

Desired properties: if and “only if"

Soundness: ✓

$$(I \perp_c J \mid K)_{\mathcal{G}} \implies (X_I \perp\!\!\!\perp X_J \mid X_K)_{\mathbb{P}} \quad \text{Niemiro and Rajkowski [2023]}$$

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1. Equivalence
2. Completeness
3. Conclusions

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- Link between the summary graph and the process

$$\mathbb{P}\big(X(t) \in F \mid X(< t) \in E\big) = \prod_{v \in \mathcal{V}} \mathbb{P}\big(X_v(t) \in F_v \mid X_{\mathrm{pa}_{\mathcal{G}}(v)}(< t) \in E_{\mathrm{pa}_{\mathcal{G}}(\mathrm{v})}\big).$$

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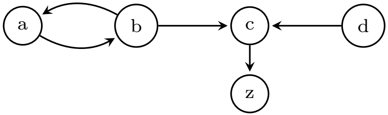

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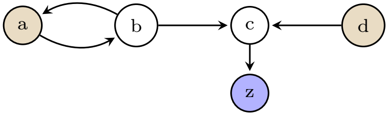

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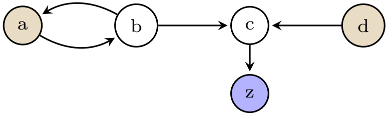

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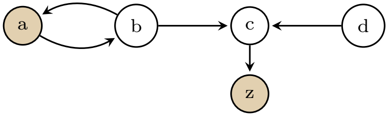

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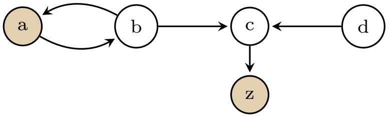

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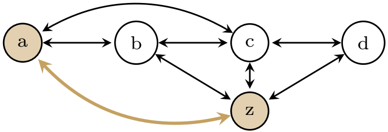

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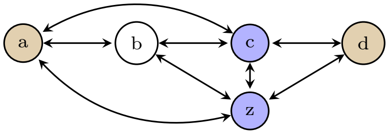

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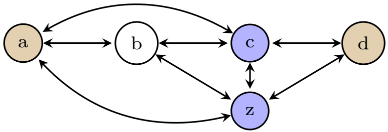

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- $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ are *c-equivalent* if $(I \perp_c J \mid K)_{\mathcal{G}_1} \iff (I \perp_c J \mid K)_{\mathcal{G}_2}$.

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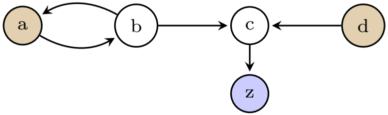
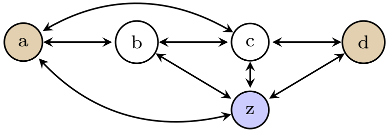
- Colliders that define c-separation are the "bottlenecks/bridges" in the trek graph





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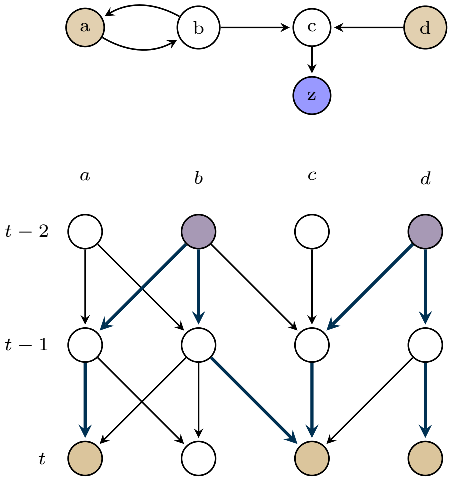

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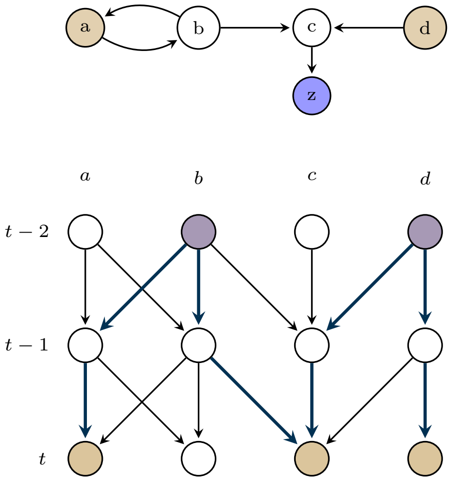

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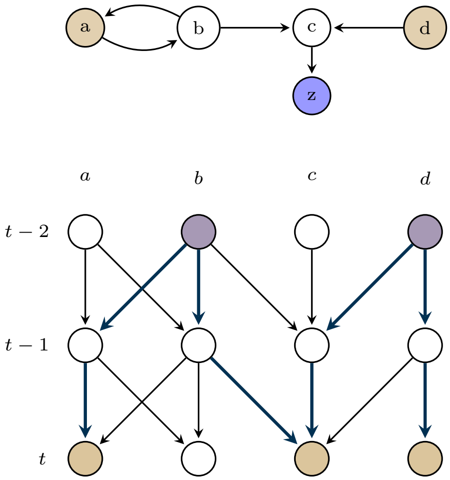

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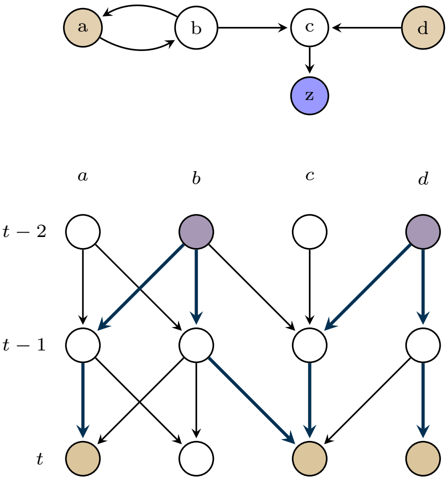
- This is c-separation translated to space-time graph when self loops are present



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- Dynamic version: $\Sigma = \sum_{k=0}^{\infty} M^k D (M^\top)^k$

Collider separation is "necessary" for conditional independence

Weak completeness

Take a summary graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and let

$$\mathcal{I}_\mathcal{G} := \left\{ \ (I, J, K) : I, J, K \subseteq \mathcal{V} \text{ pairwise disjoint and } \ (I \perp_c J \mid K)_\mathcal{G} \ \right\}.$$

Then there exists a strictly stationary stochastic process $X = \big(X_v(t)\big)_{v \in \mathcal{V}, \ t \in \mathbb{Z}}$ such that,

1. *Soundness.* For *every* triple $(I, J, K) \in \mathcal{I}_\mathcal{G}$: $X_I(t) \perp\!\!\!\perp X_J(t) \mid X_K(t)$,
2. *Weak Completeness.* For *any* triple $(I, J, K) \notin \mathcal{I}_\mathcal{G}$: $X_I(t) \not\!\perp\!\!\!\perp X_J(t) \mid X_K(t)$.

Statement

Strong completeness for Gaussian VAR(1)

Let X be a stable Gaussian VAR(1).
Let $\Theta = \{(M, D) : \rho(M) < 1,\ D \succ 0,\ M \in \mathbb{R}^{\mathcal{E}}\}$ be the open parameter domain.

Fix three pairwise disjoint spatial node sets $A, B, C \subseteq \mathcal{V}$. Then:

1. *Soundness.* If $(I \perp_c J \mid K)_{\mathcal{G}}$ then $X_I(t) \perp\!\!\!\perp X_J(t) \mid X_K(t)$,
2. *Strong completeness.* If $(I \not\perp_c J \mid K)_{\mathcal{G}}$, then $\{(M, D) \in \Theta : X_I(t) \perp\!\!\!\perp X_J(t) \mid X_K(t)\}$ has Lebesgue measure zero.

Proof: Idea from Sullivant et al. [2010] updated to a space-time trek version

Paths and the covariance matrix

- Pathwise interpretation: $(M^k D (M^\top)^k)_{ij} = \sum_{s \in \text{ancestors}} (M^k)_{is} \, D_{ss} \, (M^k)_{js}$

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$$\omega(M, D, \tau) := d_u \prod_{(k,r-1)\to(l,r)\in P_1} M_{lk} \prod_{(k,r-1)\to(l,r)\in P_2} M_{lk}.$$

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- $$\begin{aligned}\Sigma_{ij} &= \mathbf{1}_{\{i=j\}} d_i + \sum_{s\geqslant 1} \sum_{u=1}^{n} (M^s)_{iu} d_u (M^s)_{ju} \\ &= \mathbf{1}_{\{i=j\}} d_i + \sum_{s\geqslant 1} \sum_{u=1}^{n} \sum_{P_1:(u,t-s)\rightsquigarrow(i,t)} \sum_{P_2:(u,t-s)\rightsquigarrow(j,t)} \omega(M, D, (P_1, P_2))\end{aligned}$$

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Future

- Relax the Markov-order-1 assumption.
- Understand how far strong completeness extends beyond Gaussian dynamics.
- Better understanding of causality from snapshots of dynamical systems:
  - Interventions
  - Relation between time, dependence and snapshot information
  - Constrained nonstationarity

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